Does Allan Variance Determine the Spectrum?

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Abstract

The phase-noise spectrum determines the Allan variance by a well-known integral formula. It is shown here that unique inversion of this formula is not possible in general because the mapping from spectrum to Allan variance is not one-to-one. A necessary and sufficient condition for two distinct phase spectra to have the same Allan variance is given.

1 Structure Functions

This work was motivated by certain explicit and implicit claims in a 1976 paper of Lindsey and Chie [1], which uses "structure functions" to study nonstationary models of oscillator noise. To describe the setup, let $x(t)$ be a real-valued random-process model for the time deviation or phase modulation (PM) of a frequency source or pipe. It is convenient to classify these models by their "degree of nonstationarity". Let us say that a noise model $x(t)$ has *degree* $n \geq 1$ if the nth difference $\Delta_\tau^n x(t)$ (as a function of t for any fixed τ) is a stationary process, but $\Delta_\tau^{n-1} x(t)$ is nonstationary. As indicated below, the degree of the process also measures the degree of low-frequency divergence, or "redness", of the power spectrum of the process. If $x(t)$ is itself stationary we say that it has degree 0; an example is white PM noise (with a high-frequency rolloff beyond some f_h). For degree 1, $x(t)$ is nonstationary but the average frequency $\bar{y}_\tau(t) = \Delta_\tau x(t)/\tau$ is stationary; familiar examples are flicker PM (rolled off beyond f_h) and white FM. For degree 2, $\bar{y}_\tau(t)$ is no longer stationary, but its differences are stationary; examples are flicker FM and random-walk FM.

Let the PM process $x(t)$ have degree $\leq n$ (that is to say, let $x(t)$ have stationary nth differences). The covariance structure of the nth differences is specified by the function

$$D_n(t, \tau_1, \tau_2) = \operatorname{cov}\left[\Delta_{\tau_1}^n x(u+t), \Delta_{\tau_2}^n x(u)\right], \quad (1)$$

which I shall call the nth *complete* structure function. (The stationarity assumption eliminates dependence on u.) This function can be obtained from the one-sided spectral density $S_x(f)$ of the process (the PM spectrum) by integrating the transfer functions of the difference operators against it [2]:

$$D_n(t, \tau_1, \tau_2) = \int_0^\infty \cos\left(2\pi f\left[t - \frac{n}{2}(\tau_1 - \tau_2)\right]\right) \times \left[4 \sin(\pi f \tau_1) \sin(\pi f \tau_2)\right]^n S_x(f)\, df. \quad (2)$$

On the other hand, from (2) we see that the mild restriction $D_n(t, \tau, \tau)$ is the Fourier transform of $4^n \sin^{2n}(\pi f \tau) S_x(f)$ (the spectrum of the process $\Delta_\tau^n x(t)$); consequently, by Fourier inversion one can recover $S_x(f)$ for all $f > 0$, except for the possibility of unknown delta functions at integer multiples of $1/\tau$. To cover this possibility, just change τ. It follows that the nth complete structure function and the spectrum are in a one-to-one relationship.

Lindsey and Chie base their analyses on a simpler "nth structure function" of one variable, $D_n(\tau)$, given by a more severe restriction of the nth complete structure function, namely,

$$D_n(\tau) = D_n(0, \tau, \tau) = \operatorname{var}\left[\Delta_\tau^n x(t)\right] = \int_0^\infty \left[2 \sin(\pi f \tau)\right]^{2n} S_x(f)\, df. \quad (3)$$

Scaled versions of the first three $D_n(\tau)$ have been exploited by the time and frequency community to characterize and specify frequency stability for phase noises of degree 0 through 3; these "time-domain" stability measures are the average-frequency variance $\operatorname{var} \bar{y}_\tau = D_1(\tau)/\tau^2$, Allan variance $\sigma_y^2(\tau) = D_2(\tau)/(2\tau^2)$, and Hadamard variance $D_3(\tau)/(6\tau^2)$. The scaling is contrived to make all three of these equal when applied to white FM noise.

Because of the simplicity of the description of $D_n(\tau)$ and the ease of constructing statistical estimators of it, one hopes that knowledge of $D_n(\tau)$ alone is enough to determine the whole covariance structure. Unfortunately, because (3) no longer asserts an explicit Fourier-transform relationship, it is not obvious that one can go from $D_n(\tau)$ back to $S_x(f)$, nor to the complete structure function. Therefore, in order not to beg

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the question by terminology, I prefer the neutral term *nth-difference variance* for $D_n(\tau)$. Although Lindsey and Chie give two kinds of inversion formulas for $S_x(f)$ or an equivalent in terms of $D_n(\tau)$, it is not obvious that the mapping from $S_x(f)$ to $D_n(\tau)$ is even one-to-one. If the mapping is many-to-one, then any inversion formula must be at most one-sided: starting from $D_n(\tau)$ the inversion gives a spectrum that does map to $D_n(\tau)$, but it might not be the *right* spectrum. Thus, instead of trying to verify inversion formulas, I concentrate on the question of spectral uniqueness: whether or not more than one spectrum can map to the same nth-difference variance.

1.1 First-Difference Variance

For the first-difference variance, the answer to the question of spectral uniqueness has been known at least since 1940 from Kolmogorov's work on turbulence. For real-valued models $x(t)$ of degree ≤ 1, it is easy to show that $D_1(\tau)$ determines the first complete structure function. The identity

$$\begin{aligned} 2(x_1 - x_2)(x_3 - x_4) &= (x_1 - x_4)^2 - (x_1 - x_3)^2 \\ &\quad - (x_2 - x_4)^2 + (x_2 - x_3)^2 \end{aligned}$$

implies that

$$\begin{aligned} & 2\operatorname{cov}\left[x(t_1) - x(t_2), x(t_3) - x(t_4)\right] \\ = \; & D_1(t_1 - t_4) - D_1(t_2 - t_3) \\ & - D_1(t_2 - t_4) + D_1(t_2 - t_3) \end{aligned}$$

for any times t_1, t_2, t_3, t_4. Any first-difference covariance can be expressed in terms of four values of the first-difference variance, which therefore determines the complete structure function and so the spectrum. In this sense, the first-difference variance[1] $D_1(\tau)$ does deserve to be called a "structure function".

1.2 Allan Variance

The focus of this paper is the scaled version of second-difference variance called Allan variance, defined by

$$\begin{aligned} \sigma_y^2(\tau) &= \frac{1}{2\tau^2} D_2(\tau) \\ &= \frac{1}{2\tau^2} \operatorname{var}\left[x(t) - 2x(t-\tau) + x(t-2\tau)\right]. \end{aligned}$$

Devised to satisfy a need for characterizing phase noise of degree ≤ 2 [3, 4], Allan variance is the most often-used method for reducing a clock-noise time series to a statistical summary of frequency stability; its use has also spread to other fields of science as a tool for studying low-frequency spectral behavior of physical processes.

Eq. (3), with $n = 2$, gives

$$\sigma_y^2(\tau) = \frac{8}{\tau^2} \int_0^\infty \sin^4(\pi f \tau) S_x(f)\, df, \qquad (4)$$

which expresses the mapping from spectrum to Allan variance. By (4), a power-law spectrum $S_x(f) \propto f^\beta$ ($-5 < \beta < -1$) maps to the corresponding power-law Allan variance $\sigma_y^2(\tau) \propto \tau^{-3-\beta}$. In experimental practice, linear regions in the log-log plot of an estimated $\sigma_y(\tau)$ curve are associated with the corresponding spectral power-law components. Such an identification of a parametric model from observed behavior of $\sigma_y^2(\tau)$ is here called *parametric* inversion of (4) from Allan variance back to the spectrum. There is no problem with this practice if the actual PM spectrum is known to have the desired parametric form, $S_x(f) = \sum_\beta g_\beta f^\beta$ in this case.

On the other hand, the concern of the present paper is the possibility of general nonparametric inversion. Can more than one PM spectrum map via (4) to the same Allan variance?[2] Is there a general inversion formula $S_x(f)$ in terms of $\sigma_y^2(\tau)$? Van Vliet and Handel [5], regarding (4) as an integral transform that generalizes the Fourier transform, assert that $\sigma_y^2(\tau)$ does uniquely determine $S_x(f)$, and give an inversion formula involving Laplace and Mellin transforms. The principal claim of the present paper is that Allan variance does not always determine a unique PM spectrum. Moreover, the ambiguity is centered at the most interesting case, namely, flicker-FM noise, $S_x(f) \propto f^{-3}$, whose Allan variance is constant. It is shown that Allan variance is totally insensitive to a certain class of log-periodic modulations of the spectrum by octaves (see Fig. 1 for examples). Consequently, as pointed out above, an inversion algorithm for (4) must be one-sided: starting from a given Allan variance the algorithm does not necessarily arrive at the correct spectrum, but only at some spectrum with the same Allan variance.

Two main results are given below. The crux of the matter is contained in Theorem 1, which characterizes the infinite set of PM spectra whose Allan variance equals a given constant. This theorem leads immediately to Theorem 2, which gives a necessary and sufficient condition for any two spectra to have the same Allan variance. An alternate derivation of this result is

[1] sometimes denoted by $U(\tau)$ in papers on time and frequency

[2] Lowpass-filtered white PM and flicker PM do not have the same Allan variance; there is a factor of order $\ln(f_h\tau)$ between them.

carried out from the formalism of Van Vliet and Handel. Some additional argument shows that the class of PM spectra of degree-1 noises does enjoy unique inversion of (4). A proof of Theorem 1 is given as an appendix.

2 Notation and Terminology

In the mathematics that follows, which deals mainly with spectra, not with the processes themselves, a "PM spectrum" is defined to be a nonnegative measurable function[3] $S(f)$ for $f > 0$ that satisfies

$$\int_1^\infty S(f)\,df < \infty, \quad \int_0^1 S(f)\,f^{2n}df < \infty, \tag{5}$$

for some nonnegative integer n. These spectra have finite power at high frequencies and diverge in a controlled way at low frequencies. The smallest such n is called the *degree* of S, written $\deg S$. The degree of a spectrum is the same as the degree of a PM process $x(t)$ that has $S(f)$ as its spectrum. Allan variance, as given by (4) with $S_x(f)$ replaced by $S(f)$, is finite if and only if $S(f)$ has degree ≤ 2.

It is convenient to embed the set of PM spectra in a vector space of real-valued functions. A *signed PM spectrum* $\Phi(f)$ is defined to be a measurable function such that $|\Phi(f)|$ is a PM spectrum. Its degree is defined to be that of $|\Phi(f)|$. Let us extend the notion of Allan variance to a linear mapping on the subspace of signed PM spectra of degree ≤ 2 by

$$V_{\mathrm{A}}(\tau;\Phi) = \frac{8}{\tau^2}\int_0^\infty \sin^4(\pi f\tau)\,\Phi(f)\,df, \tag{6}$$

which will still be called Allan "variance" even though it can assume any real value, including zero.

3 Results for Allan Variance

The first result says that the most general PM spectrum with a constant Allan variance is obtained from a log-periodic modulation of an f^{-3} spectrum by octaves. The result is established here for signed PM spectra so that it can easily be applied to the proof of Theorem 2.

Theorem 1 *A signed PM spectrum $\Phi(f)$ has a constant Allan variance V_{A} if and only if*

$$\Phi(2f) = \frac{\Phi(f)}{8} \ a.e. \tag{7}$$

[3]This theory can also be carried out in the context of general spectral measures, which include delta functions and other singular measures. Indeed, one of the examples below consists entirely of delta functions.

In this case,

$$V_{\mathrm{A}} = 8\pi^2\int_1^2 f^2\Phi(f)\,df. \tag{8}$$

(a.e. = almost everywhere with respect to Lebesgue measure.)

Some remarks and examples follow.

a) The condition (7) on $\Phi(f)$ is equivalent to the representations

$$\Phi(f) = \frac{\phi(f)}{f^3} = \frac{\psi(\log_2 f)}{f^3} \text{ a.e.}, \tag{9}$$

where $\phi(2f) = \phi(f)$ for all f, and $\psi(x)$ is a function with period 1, integrable over a period. Then (8) becomes

$$V_{\mathrm{A}} = 8\pi^2\ln 2\int_0^1 \psi(x)\,dx. \tag{10}$$

b) The range of integration in (8) can be any octave $a < f < 2a$. That is so because the interval of integration in (10) can be replaced by any interval of length 1.

c) Any locally integrable function $\Phi(f)$ that satisfies (7) is a signed PM spectrum of degree 2, or is identically zero a.e. This can be shown by expressing the integrals of $S_x(f) = |\Phi(f)|$ in (5) as sums of integrals over octaves $[2^n, 2^{n+1}]$ for integers n.

Examples of PM spectra with the same constant Allan variance $8\pi^2\ln 2$ are shown in Fig. 1(a). The straight line is just f^{-3}. The PM spectrum $S_1(f)$ is given by

$$S_1(f) = f^{-3}\left[1 - 0.9\cos(2\pi\log_2 f)\right].$$

The series of rectangles is an approximation to the pure delta-function spectrum

$$S_0(f) = \ln 2\sum_{n=-\infty}^{\infty} 4^{-n}\delta(f - 2^n). \tag{11}$$

which lies slightly outside the mathematical framework given here. The approximating rectangles have height proportional to 8^{-n} but area proportional to 4^{-n}. Among PM spectra with constant Allan variance, this one is the most extreme[4] in that all the power in each octave is concentrated at one frequency.

The proof that $S_0(f)$ has the same constant Allan variance as f^{-3} is short and instructive. By (6),

$$V_{\mathrm{A}}\left(\frac{x}{\pi};S_0\right) = 8\pi^2\ln 2\sum_{n=-\infty}^{\infty}\frac{\sin^4(2^n x)}{4^n x^2}. \tag{12}$$

[4]but canonical in the sense that it generates all others by a logarithmic convolution operation

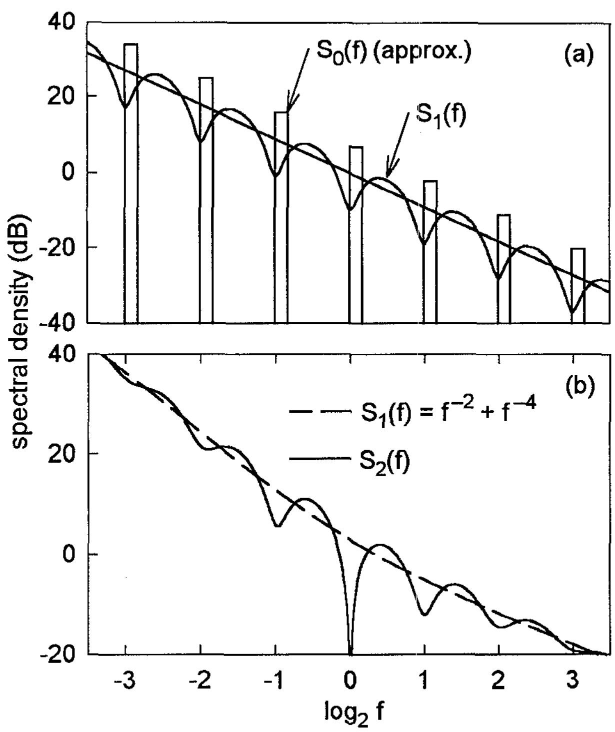

Figure 1: Examples of PM spectra with the same Allan variance. a) Examples for Theorem 1: three spectra with the same constant Allan variance. The straight line is f^{-3}. The rectangles approximate a delta-function spectrum. b) Example for Theorem 2: two spectra with the same nonconstant Allan variance.

Because of the critical identity

$$\sin^4 x = \sin^2 x - \frac{1}{4}\sin^2 2x, \tag{13}$$

the summation in (12) equals

$$\lim_{n\to-\infty}\sum_{k=n}^{\infty}\left[\frac{\sin^2(2^k x)}{4^k x^2} - \frac{\sin^2(2^{k+1}x)}{4^{k+1}x^2}\right]. \tag{14}$$

For each n, the series in (14) telescopes to the single term $4^{-n}x^{-2}\sin^2(2^n x)$, whose limit as $n \to -\infty$ is 1. Hence $V_A(\tau; S_0) = 8\pi^2 \ln 2$.

The second main result, the characterization of the spectral ambiguity of Allan variance, is an immediate corollary of Theorem 1. Two PM spectra have the same Allan variance if and only if their difference is a signed PM spectrum with zero Allan variance; thus Theorem 1 applies with $V_A = 0$.

Theorem 2 *Two PM spectra $S_1(f)$ and $S_2(f)$ of degree ≤ 2 have the same Allan variance if and only if the signed PM spectrum $\Phi(f) = S_1(f) - S_2(f)$ satisfies $\Phi(2f) = \Phi(f)/8$ a.e. and*

$$\int_1^2 f^2\Phi(f)\,df = 0.$$

Remarks (a) and (b) above hold here also; in particular, the integral of $f^2\Phi(f)$ over any octave is zero.

If $f^3 S_x(f) \geq a > 0$ for all f, then one can obtain other PM spectra with the same Allan variance as $S_x(f)$ by adding a variety of log-periodic "modulations" of form $f^{-3}\psi(\log_2 f)$ with $\psi(x+1) = \psi(x)$ and $\int_0^1 \psi(x)\,dx = 0$. This is illustrated in Fig. 1(b), which shows the two PM spectra

$$\begin{aligned} S_1(f) &= f^{-2} + f^{-4}, \\ S_2(f) &= S_1(f) - 2f^{-3}\cos(2\pi\log_2 f), \end{aligned}$$

both of which have Allan variance $2\pi^2/\tau + 8\pi^4\tau/3$. Just enough of the modulation has been added to make $S_2(1) = 0$ while keeping $S_2(f) \geq 0$. Suitably scaled in amplitude and frequency, the PM spectrum $S_1(f)$, which is just white FM plus random-walk FM, is often used as a noise model for rubidium or cesium-beam frequency standards.

3.1 Octave Variance

This name is given here to the expression

$$V_o(\tau; \Phi) = 8\pi^2 \int_{1/(4\tau)}^{1/(2\tau)} f^2\Phi(f)\,df, \tag{15}$$

which was introduced by Percival [6] as an ideal version of a bandpass variance of Rutman [7]. Again, this "variance" can assume any real value on signed PM spectra. It leads to a reformulation of Theorems 1 and 2. Since the derivative of $\int_a^{2a} f^2\Phi(f)\,df$ with respect to a equals $8a^2\Phi(2a) - a^2\Phi(a)$ a.e., it follows that $V_o(\tau; \Phi)$ is constant if and only if $\Phi(f)$ satisfies the condition (7) of Theorem 1. Thus, Theorem 1 says that a signed PM spectrum $\Phi(f)$ has a constant Allan variance if and only $\Phi(f)$ has a constant octave variance, i.e., the corresponding signed FM spectrum $4\pi^2 f^2\Phi(f)$ gives equal (signed) power to every octave $a < f < 2a$ for $a > 0$; in this case, the two variances V_A and V_o are equal. Theorem 2 says that two PM spectra of degree ≤ 2 have the same Allan variance if and only if their difference has octave variance zero, i.e., the corresponding FM spectra give the same power to every octave. The nullspaces of the V_A and V_o operators turn out to be the same.

3.2 Another Derivation of the Ambiguity

Although Van Vliet and Handel [5] claim unique inversion of Allan variance to spectrum, their method actually leads to another derivation of the nonuniqueness condition of Theorem 2. After taking the Laplace transform of both sides of (6), they solve the resulting integral equation by complex Mellin transforms. The solution for $\Phi(f)$ contains an additive term

$$\oint_C F(p) \frac{\cos(p\pi/2)}{1-2^{p-3}} \frac{dp}{\omega^p}$$

(where $\omega = 2\pi f$) that represents the general solution of the homogeneous equation, i.e., the spectral ambiguity. Here, C is a contour and $F(p)$ some analytic function in the strip $1 < \mathrm{Re}\, p < 5$. Because the integrand has a simple pole at $p_n = 3 + i2\pi n/\ln 2$ for each nonzero integer n, the sum of the residues takes the form

$$\sum_{n\neq 0} c_n \omega^{-p_n} = \omega^{-3} \sum_{n\neq 0} c_n \exp(-i2\pi n \log_2 \omega),$$

which is of form $f^{-3}\psi(\log_2 f)$ with $\psi(x)$ of period one and integral zero over a period, as specified in Theorem 2.

3.3 Stationary FM

From the results already given, one can deduce that unique inversion of the Allan variance formula (4) is indeed possible if the FM noise is stationary, i.e., if the PM noise $x(t)$ has degree ≤ 1. Suppose that $S_1(f)$ and $S_2(f)$ are *distinct* nonnegative PM spectra of degree ≤ 2 with the same Allan variance. Then both their degrees must be 2. Proof: according to Theorem 2, $S_1(f) = S_2(f) + \Phi(f)$, where $\Phi(f)$ satisfies the conditions given there. Since $S_1(f) \geq 0$, $S_2(f) \geq 0$, it follows that

$$S_1(f) \geq \Phi_+(f), \tag{16}$$

where $\Phi_+(f) = \max(\Phi(f), 0)$. Since $f^2\Phi(f)$ is not a.e. zero on an octave but integrates to zero there, the PM spectrum $\Phi_+(f)$ cannot be a.e. zero. By remark (c) following Theorem 1, $\deg \Phi_+ = 2$. By (16), $\deg S_1 = 2$. By a similar argument, $\deg S_2 = 2$. One concludes that *unique inversion of the Allan variance formula (4) is possible for PM spectra of degree* ≤ 1. Examples include power laws f^β, $-3 < \beta \leq 0$ (with a high-frequency rolloff at some f_h if $\beta \geq -1$), and integrated Lorentzians $f^{-2}\left(f^2 + f_0^2\right)^{-1}$. These processes are already characterized by the first-difference variance $D_1(\tau) = \mathrm{var}[x(t) - x(t-\tau)]$, which, as we saw, also determines the spectrum uniquely. Even so, the inversion problem is ill-posed: for example, $\sigma_y^2(\tau)$ and $D_1(\tau)$ both distinguish the flicker PM spectrum f^{-1} from the white PM spectrum f^0 by a factor of order $\ln(f_h\tau)$, which is hard to see in practice. This was the main reason for introducing the modified Allan variance [8].

4 Higher Differences

Third-difference variances have been used as stability measures in at least two contexts: 1) Hadamard variance $\mathrm{var}\left[\Delta_\tau^3 x(t)\right] / \left(6\tau^2\right)$ is used to eliminate linear frequency drift from the measurements while characterizing phase noise $x(t)$ of degree ≤ 3 [11]; 2) a continuous analog of modified Allan variance is given by $\mathrm{var}\left[\Delta_\tau^3 w(t)\right] / \left(2\tau^3\right)$, where $w(t)$, defined as $\int x(t)\,dt$, has degree ≤ 3 if $x(t)$ has degree ≤ 2 [12].

The spectral uniqueness problem for $D_3(\tau)$ leads to consideration of signed spectra of form $f^{-5}\phi(f)$, where $\phi(f)$ satisfies the functional equation

$$5\phi(f) - 32\phi(f/2) + 27\phi(f/3) = 0.$$

The solutions of this equation are far more intricate than those of the analogous equation $\phi(f) - \phi(f/2) = 0$ that arises in the second-difference case. In general, they appear to grow and oscillate explosively; I do not know whether any of them except the trivial solution $\phi(f) = \mathrm{const}$ correspond to meaningful signed spectra at all. To my knowledge, the spectral uniqueness problem for third and higher differences is open.

5 Does It Matter?

The spectral nonuniqueness results for Allan variance are given here, not to discourage the conventional use of Allan variance for analyzing time series, but simply to expose a previously unknown limitation of the technique. One can object that these results are irrelevant because the log-periodic spectral modulations that constitute the ambiguity do not arise from any known physical theory; consequently, any spectral disturbances lying in the nullspace of the integral operator given by (6) can be excluded on physical grounds. Naturally, if one of the spectra belonging to a spectrally ambiguous Allan variance has a simple parametric form, as in the examples of Fig. 1, then it is reasonable to exclude the other spectra; the example of Fig. 1(b) is intended only to show how the ambiguity works. In the general nonparametric case, one would need an objective criterion for a physically relevent spectrum in order to choose which spectrum is the right one or to decide whether a proposed inversion algorithm introduces a physically objectionable spectral disturbance.

Even if the previous objection is granted, I argue that users of Allan variance still ought to be aware of the facts that are proved here. First, since other researchers have asserted unique invertibility, the record needs to be set straight. The operation (6) ought not to be regarded as an integral transform that extends the Fourier transform. Second, the outputs of artificial $1/f$ noise generators built from ladders of first-order analog or digital filters [9, 10] have just this kind of modulated spectrum, although the frequency ratio of the ladder is not usually designed to be 2, and there are only a few filter stages in practice. Nevertheless, it is unsafe to use flatness of Allan variance of the integrated output as a test of the spectral accuracy of such generators. Finally, these results put a fundamental limitation on what can be learned about a noise process from examination of its Allan variance, which, in general, does not completely characterize the covariance properties of the noise. Although the Allan-variance statistic remains useful for revealing broad spectral trends, the extraction of spectral details by this means is difficult, if not impossible.

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References

[1] W. C. Lindsey and C. M. Chie, "Theory of oscillator instability based upon structure functions", *Proc. IEEE*, vol. 64, pp. 1652–1666, 1976.

[2] A. M. Yaglom, "Correlation theory of processes with random stationary nth increments", in *American Mathematical Society Translations*, series 2, vol. 8, pp. 87–141, 1958.

[3] D. W. Allan, "Statistics of atomic frequency standards", *Proc. IEEE*, vol. 54, pp. 221–230, 1966.

[4] J. A. Barnes, "Atomic timekeeping and the statistics of precision signal generators, *Proc. IEEE*, vol. 54, pp. 207–220, 1966.

[5] C. M. Van Vliet and P. H. Handel, "A new transform theorem for stochastic processes with special application to counting statistics", *Physica A*, vol. 113, pp. 261–276, 1982.

[6] D. B. Percival, "Characterization of frequency stability: frequency-domain estimation of stability measures, *Proc. IEEE*, vol. 79, pp. 961–972, 1991.

[7] J. Rutman, "Characterization of phase and frequency instabilities in precision frequency standards: fifteen years of progress", *Proc. IEEE,* vol. 66, pp. 1048–1075, 1978.

[8] D. W. Allan and J. A. Barnes, "A modified 'Allan variance' with increased oscillator characterization ability", *Proc. 35th Ann. Frequency Control Symposium*, pp. 470–474, 1981.

[9] J. A. Barnes, *The generation and recognition of flicker noise,* NBS Report 9284, National Bureau of Standards, Boulder, 1967.

[10] J. A. Barnes and S. Jarvis Jr., *Efficient numerical and analog modeling of flicker noise processes,* NBS Tech. Note 604, National Bureau of Standards, Boulder, 1971.

[11] S. T. Hutsell, "Relating the Hadamard variance to MCS Kalman filter clock estimation", *Proc. 27th Ann. Precise Time and Time Interval (PTTI) Applications and Planning Meeting*, pp. 291–302, 1995.

[12] C. A. Greenhall, "The third-difference approach to modified Allan variance", *IEEE Trans. Instrum. Meas.*, vol. 46, pp. 696–703, 1997.

[13] C. A. Greenhall, "A structure function representation theorem with applications to frequency stability estimation", *IEEE Trans. Instrum. Meas.*, vol. IM-32, pp. 364–370, 1983.

A Proof of Theorem 1

The "if" part of Theorem 1, and the formula (8) for V_{A}, can be proved by generalizing the derivation of the Allan variance of the delta-function spectrum (11). Assume merely that $\Phi(f)$ is a locally integrable function satisfying (7). By Remark (b) following Theorem 1, $\Phi(f)$ is a signed PM spectrum of degree 2 (or vanishes a.e.). Then, for any $\tau > 0$, $\sin^4(\pi f \tau)\,\Phi(f)$ is integrable, and $V_{\mathrm{A}}(\tau;\Phi) = 8\tau^{-2}\lim_{n\to-\infty} I_n(\tau)$, where

$$I_n(\tau) = \int_{2^n}^{\infty} \sin^4(\pi f \tau)\,\Phi(f)\,df.$$

Use the trigonometric identity (13) to express $I_n(\tau)$ as the difference of two integrals, in the second of which make the change of variable $f' = 2f$ and apply (7). The two integrals, now having the same integrand, recombine to give

$$I_n(\tau) = \int_{2^n}^{2^{n+1}} \sin^2(\pi f \tau)\,\Phi(f)\,df,$$

which, via another change of variable and (7) again, gives

$$I_n(\tau) = \pi^2\tau^2 \int_1^2 \left[\frac{\sin(2^n \pi\tau f)}{2^n \pi \tau f}\right]^2 f^2 \Phi(f)\,df.$$

As $n \to -\infty$, the sinc function tends uniformly to 1, and $I_n(\tau)$ therefore tends to $\pi^2\tau^2 \int_1^2 f^2\Phi(f)\,df$.

The proof of the "only if" part of Theorem 1 depends on properties of the *generalized autocovariance* (gacv) function [13], defined for signed PM spectra of degree d by

$$R(t;\Phi) = \int_0^\infty C_d(2\pi f, t)\,\Phi(f)\,df, \qquad (17)$$

where

$$C_d(\omega, t) = \cos(\omega t) - \frac{1}{1+\omega^{2d}} \sum_{j=0}^{d-1} \frac{(-1)^j (\omega t)^{2j}}{(2j)!}. \qquad (18)$$

If $d = 0$, then the sum in (18) is omitted, and $R(t;\Phi)$ is just the cosine transform of $\Phi(f)$. The integral in (17) exists absolutely because of (5) and (23) below.

Some facts about the gacv will be developed in a sequence of propositions, the first of which deals with scaling and linearity.

Proposition 1 *Let Φ, Φ_1, Φ_2 be signed PM spectra, a, a_1, a_2 real numbers, $a > 0$. The expressions*

$$aR(t;\Phi(a\,\cdot)) - R(t/a;\Phi), \qquad (19)$$

$$R(t; a_1\Phi_1 + a_2\Phi_2) - a_1 R(t;\Phi_1) - a_2 R(t;\Phi_2) \qquad (20)$$

are polynomials, where $\Phi(a\,\cdot)$ denotes the function $f \to \Phi(af)$.

This can be shown by straightforward manipulations of (17). It is necessary to observe that if $\deg\Phi = d_1 < d$, then the right side of (17) differs from $R(t;\Phi)$ by a polynomial. Therefore, when evaluating the members of (20), one can take $d = \max(\deg\Phi_1, \deg\Phi_2)$ in (17).

It is now to be shown that R and Φ form a generalized Fourier-transform pair with respect to a certain space of test functions. For this purpose, a "test function" is defined to be a complex-valued function $\nu(t)$, defined for all real t, such that $\nu(t)$ is the inverse Fourier transform of a function $\hat{\nu}(f)$ that is infinitely differentiable and vanishes outside some closed, bounded subinterval of the positive real line. Let $\nu(t)$ be a test function. Then $\nu(t)$ is bounded and $\int_{-\infty}^{\infty} \nu(t)\,dt = \hat{\nu}(0) = 0$. Integrating the inverse Fourier relationship repeatedly by parts, one finds that $t^n \nu(t)$ is also a test function for any positive integer n. Therefore, for any polynomial $p(t)$, $p(t)\,\nu(t)$ is integrable and $\int_{-\infty}^{\infty} p(t)\,\nu(t)\,dt = 0$, i.e, test functions "kill polynomials".

Proposition 2 *If $\Phi(f)$ is a signed PM spectrum, then*

$$\int_{-\infty}^{\infty} \nu(t)\,R(t;\Phi)\,dt = \frac{1}{2}\int_0^\infty \hat{\nu}(f)\,\Phi(f)\,df \qquad (21)$$

for all test functions $\nu(t)$.

Proof. Let $\deg\Phi = d$. By (17), the left side of (21) equals

$$\int_{-\infty}^{\infty} dt\ \nu(t) \int_0^\infty df\ \Phi(f)\,C_d(2\pi f, t). \qquad (22)$$

Writing

$$C_d(\omega,t) = \frac{1}{1+\omega^{2d}}\left[\cos\omega t - \sum_{j=0}^{d-1}\frac{(-1)^j(\omega t)^{2j}}{(2j)!}\right] + \cos\omega t \frac{\omega^{2d}}{1+\omega^{2d}},$$

one sees from Taylor's formula with remainder that

$$|C_d(\omega,t)| \le \frac{\omega^{2d}}{1+\omega^{2d}}\left(\frac{t^{2d}}{(2d)!}+1\right); \qquad (23)$$

consequently, the iterated integral (22), with the integrands replaced by their absolute values, is bounded by

$$\int_{-\infty}^{\infty} dt\,|\nu(t)|\left(\frac{t^{2d}}{(2d)!}+1\right)\int_0^\infty \frac{d\omega}{2\pi}\left|\Phi\left(\frac{\omega}{2\pi}\right)\right|\frac{\omega^{2d}}{1+\omega^{2d}},$$

which is finite. By Fubini's theorem (the integrand being jointly measurable), the integral (22) exists and the order of integration can be interchanged, giving

$$\int_{-\infty}^{\infty} \nu(t)\,R(t;\mu)\,dt = \int_0^\infty df\ \Phi(f)\int_{-\infty}^{\infty} dt\ \nu(t)\,C_d(2\pi f, t).$$

Because $\nu(t)$ kills polynomials,

$$\int_{-\infty}^{\infty} \nu(t)\,C_d(2\pi f,t)\,dt = \int_{-\infty}^{\infty} \nu(t)\cos(2\pi f t)\,dt$$

$$= \frac{1}{2}\left[\hat{\nu}(f) + \hat{\nu}(-f)\right] = \frac{1}{2}\hat{\nu}(f)$$

for $f > 0$. ■

Proposition 3 *If $\Phi(f)$ is a signed PM spectrum for which $R(t;\Phi)$ is a polynomial, then $\Phi(f) = 0$ a.e. (and hence $R(t;\Phi)$ is actually zero).*

Proof. If $R(t;\Phi)$ is a polynomial, then it is killed by all test functions $\nu(t)$. By Proposition 2, $\Phi(f)$ is orthogonal to all the test-function transforms $\hat{\nu}(f)$. Because the indicator function of any open interval $]a,b[$, where $0 < a < b < \infty$, is the limit of an increasing sequence of such transforms, it follows that $\Phi(f)$ integrates to zero over all such intervals, and therefore vanishes a.e. ■

The last proposition gives a formula for Allan variance in terms of gacv.

Proposition 4 *If $\Phi(f)$ is a signed PM spectrum of degree ≤ 2 with gacv $R(t)$, then*

$$\tau^2 V_{\mathrm{A}}(\tau;\Phi) = 3R(0) - 4R(\tau) + R(2\tau). \tag{24}$$

Proof. Let Δ_τ^2 be the backward second-difference operator: $\Delta_\tau^2 x(t) = x(t) - 2x(t-\tau) + x(t-2\tau)$ for any function $x(t)$. This operator kills polynomials of degree ≤ 1 and reduces the degree of other polynomials by 2; thus, the product operator $\Lambda = \Delta_\tau^2 \Delta_{-\tau}^2$ kills polynomials of degree ≤ 3. Applying Λ to both sides of (18) as functions of t for fixed $\omega = 2\pi f$ and $d \leq 2$ gives

$$\Lambda C_d(2\pi f, t) = 16\sin^4(\pi f\tau)\cos(2\pi f t).$$

Finally, applying Λ to both sides of (17) and setting $t = 0$ gives (in view of (6))

$$\begin{aligned} & R(-2\tau) - 4R(-\tau) + 6R(0) - 4R(\tau) + R(2\tau) \\ = \; & 2\tau^2 V_{\mathrm{A}}(\tau;\Phi), \end{aligned}$$

which is the same as (24) because $R(t;\Phi)$ is even. ■

Although the formula (24) is easy to derive from (**??**) for a stationary process with autocovariance $R(t)$, the present theory applies to a process with stationary second differences and gacv $R(t)$.

The proof of Theorem 1 can now be completed. Assume that $V_{\mathrm{A}}(\tau;\Phi)$ is constant. Then $\deg\Phi \leq 2$. Let its gacv be $R(t)$. From (24) one sees that $R(2t) - 4R(t)$ is a polynomial. By Proposition 1, the gacv of $\Phi(f/2) - 8\Phi(f)$ equals $2R(2t) - 8R(t)$ plus a polynomial, and is therefore also a polynomial. By Proposition 3, $\Phi(f/2) - 8\Phi(f) = 0$ a.e.